

# A Minimum-Error Equal-Area Pseudocylindrical Map Projection

R.E. Deakin

**ABSTRACT.** *The point pole of a pseudocylindrical map projection may be expanded to a line to alleviate distortions in the map at high latitudes. The ratio between the length of the pole line and the length of the equator may be determined so as to give a minimum-error pseudocylindrical map projection. A variation of the minimum-error technique, as proposed by Sir George Airy, is applied to the sinusoidal projection to demonstrate the method.*

**KEYWORDS:** *map projection, pseudocylindrical, minimum error, equal area, sinusoidal.*

## Introduction

Pseudocylindrical map projections may be classed as those projections whose parallels of latitude are straight lines, and meridians of longitude are equally spaced curved lines. They have similar characteristics to cylindrical projections, except for the curved meridians, hence the name pseudocylindrical. Two common pseudocylindrical projections, the sinusoidal and Eckert's No. 6, whose meridians are sine curves, are shown in Figures 1 and 2.

The curved meridians of pseudocylindrical projections give the round-earth effect to maps and allow for some flexibility in minimizing distortions in higher latitudes. Various other types of curves have been chosen for the meridians, and, as well as the sine curves already mentioned, ellipses, tangent curves, parabolic, hyperbolic, cubic, and quartic curves have been used by various authors.

In a comparison of pseudocylindrical map projections, Snyder (1977) notes that there are at least 80

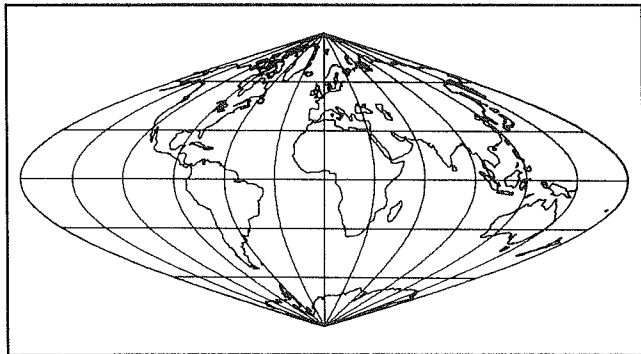


Figure 1. Sinusoidal projection.

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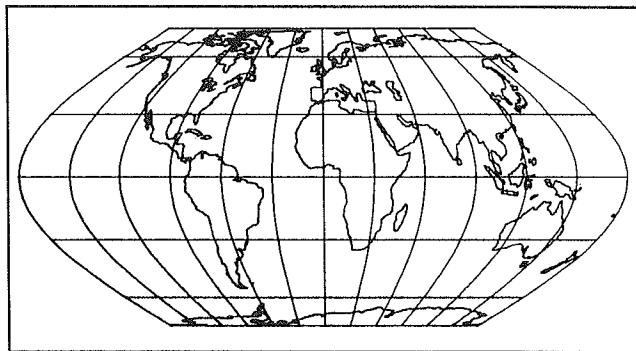


Figure 2. Eckert's No. 6 Projection.

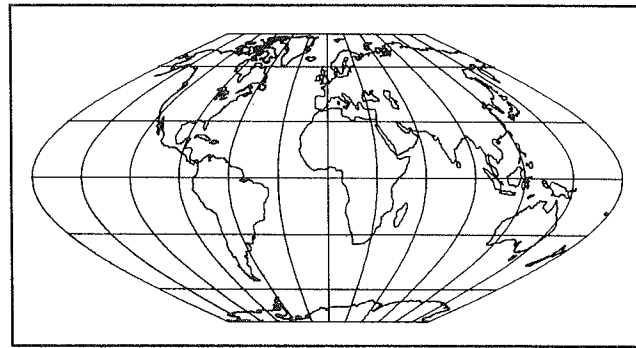


Figure 3. Thomas and McBryde's No. 3 Projection.

published projections with straight parallels and curved meridians, about 40 of which are equal area, and about 20 of which have equidistant parallels. Inspection of these projections indicates that they may be grouped in the following ways: (1) according to the particular properties of the projection (e.g., equal area or other useful property), (2) according to the type of curves used for the projected meridians, or (3) according to the type of pole, which may be a point or a line.

Using these broad classifications, Eckert's No. 6 Projection and McBryde and Thomas' No. 3 Projec-

tion (Figures 2 and 3) could be described as equal-area sinusoidal projections with pole lines equal to one-half and one-third the length of the equator, respectively.

It is interesting to note that there are no conformal pseudocylindrical projections, since the curved meridians cannot satisfy the conformal condition of intersecting all parallels (which are straight by definition) at right angles. No such restriction applies to the property of equivalence of areas, and, hence, there are numerous equal-area pseudocylindrical projections.

In a study of world map projections used for statistical purposes, McBryde and Thomas (1949) showed that there is a general family of equal-area pseudocylindrical map projections (showing the pole as a straight line somewhat shorter than the equator), which may be derived from their parent projections, whose curved meridians meet at a point pole. The expansion of the pole from a point to a line is useful in alleviating angular and scale distortions in higher latitudes, which can be seen in comparing Eckert's No. 6 Projection with its parent projection, the sinusoidal.

In a particular class of pseudocylindrical map projections, say equal-area sinusoidal with a pole line, the amount of distortion (linear and angular) is directly related to two quantities: (1) the axes ratio, which is the ratio between the lengths of the equator and the central meridian, and (2) the pole/equator ratio, which is the ratio between the lengths of the pole line and equator.

Figures 5, 6, and 7 show quadrants of equal-area pseudocylindrical graticules. In each figure, the axes ratio is one-half, while the pole/equator ratio varies from zero to one-half.

Tissot's indicatrix ellipse is plotted at selected graticule intersections, and gives a graphic indication of the variation of distortion with the change in the pole/equator ratio.

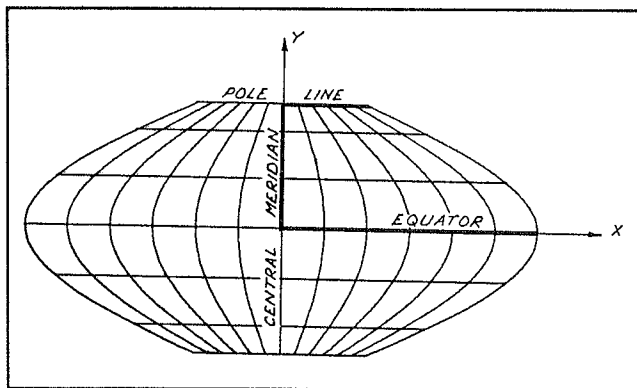


Figure 4. McBryde and Thomas' No. 3 Projection (equal-area pseudocylindrical projection graticule; axes ratio = 1/2; pole/equator ratio = 1/3; graticule interval, 30 degrees).

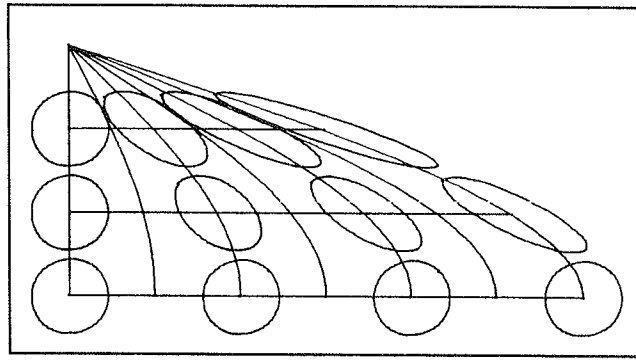


Figure 5. Pole/equator ratio = zero.

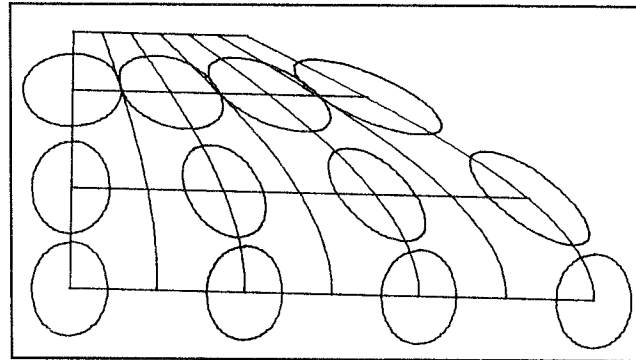


Figure 6. Pole/equator ratio = one-third.

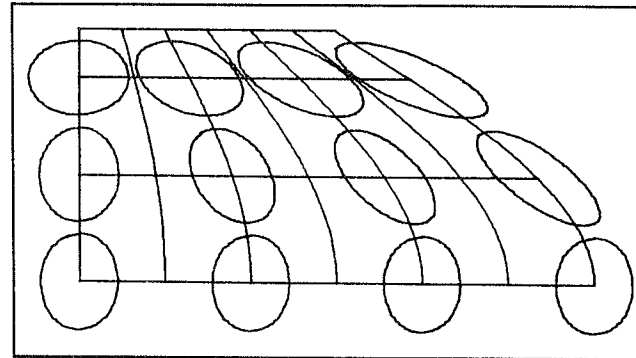


Figure 7. Pole/equator ratio = one-half.

In most pseudocylindrical map projections, the axes ratio is set to one-half, which reflects the true relationship between the central meridian and the equator on a spherical earth, and, hence, the distortions in a particular class of projections can be directly related to the pole/equator ratio.

This paper will propose a method of quantifying distortions in pseudocylindrical map projections and determine the pole/equator ratio for a minimum-error sinusoidal pseudocylindrical map projection.

### Minimum-Error Map Projection Functions

Sir George Airy (1861) proposed a method to determine projection constants, such that the sum of the squares of the scale errors in the principal directions summed for every point on the map is a minimum.

Airy called his method "Balance of Errors," and applied it to an azimuthal projection. Unfortunately, Airy made an incorrect assumption in determining the projection constants, and the benefits of his projection over other common projections of that era were minimal. Airy's unfortunate error was corrected by others a short time later, but even though his technique led to significant improvement of projections, the computational processes required were a deterrent.

A.E. Young (1920) extended Airy's method to the general conical projection, and demonstrated how the technique could be used to obtain the minimum-error projection of a particular class of projections, as well as enabling the errors or distortions to be quantified. The minimum-error function used by Young can be derived in the following manner:

1. Tissot showed that an infinitesimal unit circle on the surface of the earth will be projected as an ellipse on the map projection, and that the lengths of the semi-axes of this indicatrix ellipse are the scales in the principal directions.
2. If  $a$  and  $b$  are the lengths of the major and minor semi-axes of Tissot's indicatrix, respectively, then  $(1 - a)$  and  $(1 - b)$  are scale errors.
3. The lengths of the axes  $a$  and  $b$  are functions of latitude ( $\phi$ ) and longitude ( $\lambda$ ), and, hence, the sum of the squares of the scale errors may be represented by the function

$$f(\phi, \lambda) = [(1 - a)^2 + (1 - b)^2].$$

4. Summation over the surface of the sphere, leads to the integral

$$\int f(\phi, \lambda) da,$$

where  $da$  is the elemental area on the sphere of radius  $R$

$$da = R^2 \cos \phi d\phi d\lambda.$$

5. The function to be minimized becomes

$$Z = \int \int f(\phi, \lambda) R^2 \cos \phi d\phi d\lambda,$$

and, with  $R$  as unity, may be written as

$$Z = \int_{\lambda_1}^{\lambda_2} \int_{\phi_1}^{\phi_2} [(1 - a)^2 + (1 - b)^2] \cos \phi d\phi d\lambda, \quad (1)$$

with the integral limits chosen such that the function  $Z$  can be determined for desired portions of the map.

It will be shown in the following sections that the Cartesian equations for a general pseudocylindrical

map projection contain a variable directly related to the pole/equator ratio. Now,  $a$  and  $b$ , the maximum and minimum scales, respectively, can be obtained from the differentials of the projection equations, and, hence, the function  $Z$ , given by (1), is related to the pole/equator ratio. Evaluating  $Z$  for different pole/equator ratios will enable a minimum sum of squares to be determined, corresponding to a certain pole/equator ratio.

## The General Pseudocylindrical Equal-Area Projection

McBryde and Thomas (1949, 13) derived the general form of the equal-area pseudocylindrical projection as

$$\begin{aligned} x &= \frac{R \lambda}{M n} \left( k + \frac{\cos \alpha}{f'(\alpha)} \right) \\ y &= R M f(\alpha) \\ n \sin \phi &= k f(\alpha) + \sin \alpha \\ n &= k f\left(\frac{\pi}{2}\right) + 1, \end{aligned} \quad (2)$$

and arbitrary constants  $M$  and  $k$  are found by setting desired ratios for the projected lengths of the equator, central meridian, and pole. These are

*Axes ratio*

$$\frac{y_0}{x_0} = \frac{n M^2 f\left(\frac{\pi}{2}\right) f'(0)}{\pi (k f'(0) + 1)}, \quad (3)$$

where  $y_0$  and  $x_0$  are half the projected lengths of the central meridian and the equator, respectively.

*Pole/equator ratio*

$$\frac{x_p}{x_e} = \frac{k f'(0)}{k f'(0) + 1}, \quad (4)$$

where  $x_p$  and  $x_e$  are half the projected lengths of the pole and equator, respectively.

It should be noted that  $x_0$  is equivalent to  $x_e$ , and this length, together with the lengths  $y_0$  and  $x_p$ , are shown as heavy lines on Figure 4. In equations (2), the following conventions apply:  $x, y$  are Cartesian coordinates with the origin at the intersection of the projected equator and central meridian of the map (Figure 4);  $\phi, \lambda$  are latitude and longitude, respectively, with  $\lambda$  being the longitude difference from the central meridian;  $R$  is the radius of the spherical earth;  $\alpha$  is a parameter having similar characteristics to the latitude, and taking all values of the latitude such that  $f(\alpha) = f(\phi)$ , where the general notation  $f(\alpha)$  signifies a function of the parameter  $\alpha$ ;  $f'(\alpha)$  is the derivative

of that function and is never zero by definition, so that the quotient in the first of equations (2) is definite;  $M$  is an arbitrary constant related to the axes ratio;  $k$  is an arbitrary constant related to the pole/equator ratio; and  $n$  is a constant related to  $k$  that enables the parameter  $\alpha$  to take all the values of  $\phi$ .

Also, in equations (2), the relationship between  $n$ ,  $\phi$ ,  $k$ , and  $\alpha$  has been determined so as to maintain the equal-area property when the parent projection is modified by the introduction of a pole line.

The expression  $f\left(\frac{\pi}{2}\right)$  means the function of  $\alpha$  evaluated when  $\alpha = \frac{\pi}{2}$ , and  $f'(0)$  means the derivative of the function evaluated at  $\alpha = 0$ .

### The Sinusoidal Pseudocylindrical Equal-Area Projection

The sinusoidal projection, shown in Figure 1, has the equations

$$\begin{aligned} x &= R \lambda \cos \phi \\ y &= R \phi. \end{aligned} \quad (5)$$

Since  $f(\alpha) = f(\phi)$ , replacing  $\phi$  with  $\alpha$  gives equations with the same parameters as (2)

$$\begin{aligned} x &= R \lambda \cos \alpha \\ y &= R \alpha, \end{aligned}$$

and  $f(\alpha) = \alpha$ , therefore,  $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ , and  $f'(\alpha) = 1$

for all values of  $\alpha$  including  $\alpha = 0$ , hence,  $f'(0) = 1$ .

Substituting these expressions into equations (2), (3), and (4) gives the equations for a family of equal-area sinusoidal pseudocylindrical projections as

$$\begin{aligned} x &= \frac{R \lambda}{M n} (k + \cos \alpha) \\ y &= R M \alpha \\ n \sin \phi &= k \alpha + \sin \alpha, \end{aligned}$$

where

$$n = \frac{k \pi}{2} + 1$$

$$\frac{x_p}{x_e} = \frac{k}{k+1} = \text{the pole/equator ratio}$$

$$\frac{y_0}{x_0} = \frac{n M^2}{2(k+1)} = \text{the axes ratio.} \quad (6)$$

An actual user of equations (6) will need to calculate values for  $k$ ,  $n$ , and  $M$  for adopted pole/equator and axes ratios, and more useful working formulas may

be expressed as

$$\text{Pole/equator ratio} \quad p = \frac{x_p}{x_e},$$

$$\text{then} \quad k = \frac{p}{1-p}.$$

$$\text{Axes ratio} \quad r = \frac{y_0}{x_0},$$

$$\text{then} \quad M = \sqrt{\frac{2 r (k+1)}{n}} \quad (6a)$$

Using equations (6), the equal-area sinusoidal pseudocylindrical projections in Table 1 can be specified.

### Evaluation of the Minimum-Error Function

The error function  $Z$ , given by equation (1), contains the variables  $a$  and  $b$ , the maximum and minimum scales, respectively. These can be computed for any point on a map projection using the Gaussian Fundamental Quantities  $E$ ,  $F$ , and  $G$ , and related quantities  $E'$ ,  $F'$ , and  $G'$  given in Lauf (1983, 74) as

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 & E' &= \frac{E}{R^2} \\ F &= \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \lambda} & F' &= \frac{F}{R^2 \cos \phi} \\ G &= \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 & G' &= \frac{G}{R^2 \cos^2 \phi}, \end{aligned} \quad (7)$$

$a$  and  $b$  are given by

$$a^2 = \frac{(E' + G' + W)}{2}$$

$$b^2 = \frac{(E' + G' - W)}{2},$$

Table 1.

Projection	Pole/ Equator Ratio	Axes Ratio	$k$	$n$	$M$
Sinusoidal	0	1/2	0	1	1
Eckert No. 6	1/2	1/2	1	$\frac{\pi + 2}{2}$	$\frac{2}{\sqrt{\pi + 2}}$
McBryde/ Thomas No. 3	1/3	1/2	1/2	$\frac{\pi + 4}{4}$	$\sqrt{\frac{6}{\pi + 4}}$

where

$$W^2 = (E' - G')^2 + 4F'^2. \quad (8)$$

The partial derivatives in equations (7) can be obtained from the general projection equations (6) as

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= \frac{-R \lambda \cos \phi \sin \alpha}{M(k + \cos \alpha)} \\ \frac{\partial y}{\partial \phi} &= \frac{R M n \cos \phi}{k + \cos \alpha} \\ \frac{\partial x}{\partial \lambda} &= \frac{R(k + \cos \alpha)}{M n} \\ \frac{\partial y}{\partial \lambda} &= 0. \end{aligned} \quad (9)$$

Inspection of equations (7), (8), and (9) shows that for particular values of  $\phi$  and  $\alpha$ , the scales  $a$  and  $b$  are functions of variables  $M$  and  $k$ . For many pseudocylindrical projections, the axes ratio is one-half, which reflects the true relationship between the lengths of the equator and the central meridian on the spherical earth. Adopting a particular ratio fixes  $M$  as a constant, and causes the value of the error function  $Z$  to be dependent only upon the variable  $k$ .

Because of the symmetry, it is only necessary to evaluate the error function over one-quarter of the projection. Thus, the longitude limits can be taken as  $\lambda_1 = 0^\circ$  and  $\lambda_2 = 180^\circ$ , and the latitude limits as  $\phi_1 = 0^\circ$  and  $\phi_2 = 80^\circ$ . The latitude limit of  $80^\circ$  covers all the habitable land areas of the earth.

With the axes ratio fixed at one-half, the minimum-error function  $Z$  was solved for successive values of  $k$  by a computer program using an IMSL (International Mathematics and Statistics Library) Gaussian Quadrature procedure. Inspection of the results shown

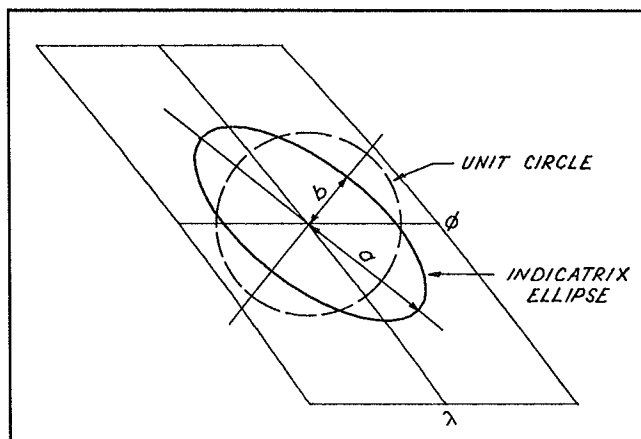


Figure 8. Diagrammatic representation of the unit circle and the indicatrix ellipse on a pseudocylindrical map projection.

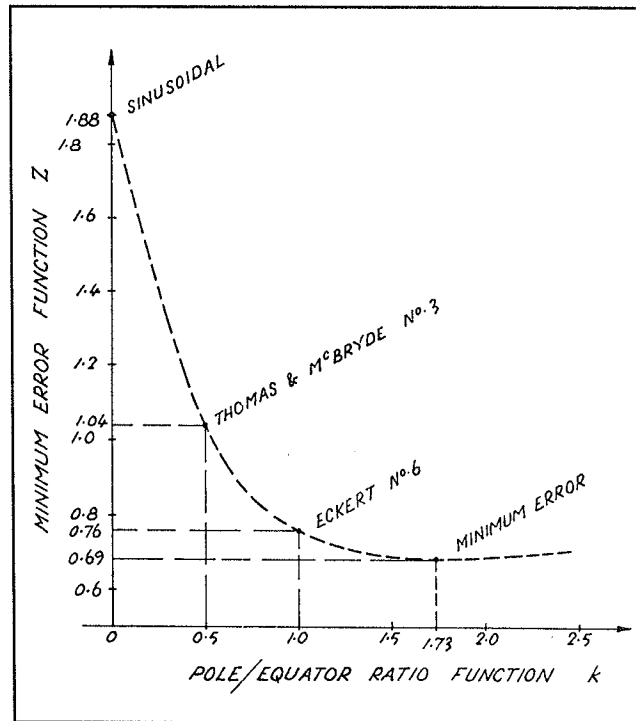


Figure 9. Plot of error function  $Z$  versus pole/equator function  $k$ .

in Figure 9 leads to a value of  $k = 1.73$  for a minimum  $Z$  value.

A tabulation of the results (Table 2) shows the minimum-error sinusoidal equal-area pseudocylindrical projection using the above basis to be a significant improvement over existing projections of the same class.

Table 1 shows  $k$ , the pole/equator ratio, the value of the minimum-error function  $Z$ , and the percentage change for four equal-area sinusoidal pseudocylindrical projections. The percentage-change column reflects the relative increase in the error function  $Z$ .

It should be noted that the axes ratio is one-half for the projections tabulated previously, and variation of this ratio will lead to other minimum-error projections.

## Conclusion

It has been demonstrated that the method of minimizing the sum of squares of scale errors, as proposed by Airy and used by Young, can be adapted to the analysis of pseudocylindrical projections, both for determining minimum-error projection constants and also for quantifying various projections.

The method could be applied to other classes of pseudocylindrical projections, such as those with elliptical, parabolic, or polynomial meridian curves, as well as varying axes ratios.

Inspection of Figure 9 shows that the value of the minimum-error function  $Z$  decreases fairly rapidly as the pole is expanded to a pole/equator ratio of ap-

**Table 2.**

k	Pole/Equator ratio	Minimum-Error Function Z	Projection Name	% change
0	0	1.879623	Sinusoidal	174
0.5	1/3	1.041670	McBryde/Thomas No. 3	52
1.0	1/2	0.757166	Eckert No. 6	10
1.73	1/1.58	0.686674	Minimum Error	0

proximately 1/2 (k = 1.0). It then decreases slowly to a minimum around 1.7, and increases very slowly thereafter. The minimum value of the error function will change for varying axes ratios and different limits of latitude and longitude, and, thus, it would be possible to deduce a pole/equator ratio for a minimum-error projection of a particular zone of the earth.

### Appendix A

Worked example – Minimum-Error Sinusoidal Pseudocylindrical Equal-Area Projection

Point Delhi (India)  $\phi = 28^{\circ}38' N.$   
 $\lambda = 77^{\circ}17' E.$

Formulas Equations (6) and (6a).

$$x = \frac{R \lambda}{M n} (k + \cos \alpha)$$

$$y = R M \alpha$$

$$n \sin \phi = k \alpha + \sin \alpha,$$

where

$$n = \frac{k \pi}{2} + 1$$

$$\frac{x_p}{x_e} = \frac{k}{k + 1} = \text{the pole/equator ratio}$$

$$\frac{y_0}{x_0} = \frac{n M^2}{2 (k + 1)} = \text{the axes ratio.}$$

Let  $p = \frac{x_p}{x_e}$  then  $k = \frac{p}{1 - p},$

and  $r = \frac{y_0}{x_0}$  then  $M = \sqrt{\frac{2 r (k + 1)}{n}}.$

For a pole/equator ratio,  $p = 1/1.58$   $k = 1.72413793$   
 $n = 3.70826953.$

For an axes ratio,  $r = 1/2$   $M = 0.85709488.$

To evaluate the parameter  $\alpha$ , the implicit equation

$$n \sin \phi = k \alpha + \sin \alpha$$

can be solved by Newton's method of iteration which is given by the formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)},$$

where

- $\alpha_n$  is the nth (present) value of the variable,
- $\alpha_{n+1}$  is the next (n + 1) value of the variable,
- $f(\alpha_n)$  is the nth value of the function,
- $f'(\alpha_n)$  is the nth value of the derivative of the function,

and

$$f(\alpha) = k \alpha + \sin \alpha - n \sin \phi$$

$$f'(\alpha) = k + \cos \alpha.$$

Using the value of the latitude ( $\phi$ ) as the initial value of the parameter  $\alpha$ , the iterative solution using Newton's method converges rapidly and is set out in the table below.

Iteration number n	$\alpha_n$	$\frac{f(\alpha_n)}{f'(\alpha_n)}$	$\alpha_{n+1}$
1	28°38'	$\frac{-0.43617877}{2.60184227}$	38°14'18.70"
2	38°14'18.70"	$\frac{-0.00740622}{2.50957880}$	38°24'27.43"
3	38°24'27.43"	$\frac{-0.00000270}{2.50774879}$	38°24'27.65"

Using the values of k, n, M, and  $\alpha$  calculated above, the Cartesian coordinates of Delhi on a projection whose principal scale is 1:200 million are:

$$x = 33.90 \text{ mm.}$$

$$y = 18.30 \text{ mm.} \quad (R = 6371 \text{ km}),$$

with the semi-axis lengths

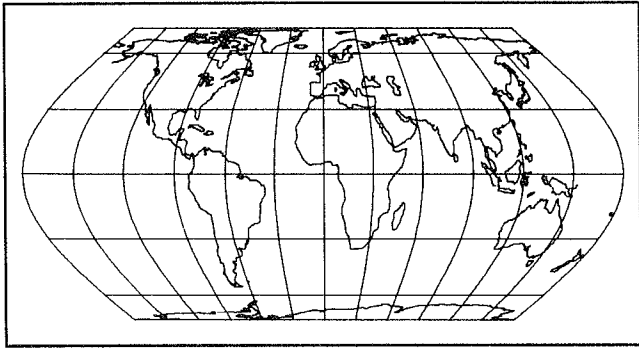
$$\text{Equator} = 85.77 \text{ mm.}$$

$$\text{Central Meridian} = 42.89 \text{ mm.}$$

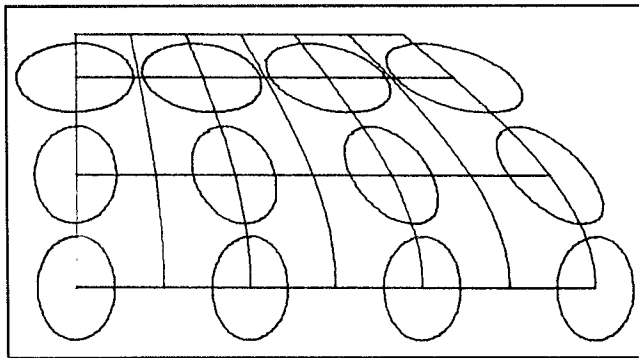
Figure A1 shows Delhi plotted on a portion of the map projection, plotted at a smaller scale.

### Appendix B

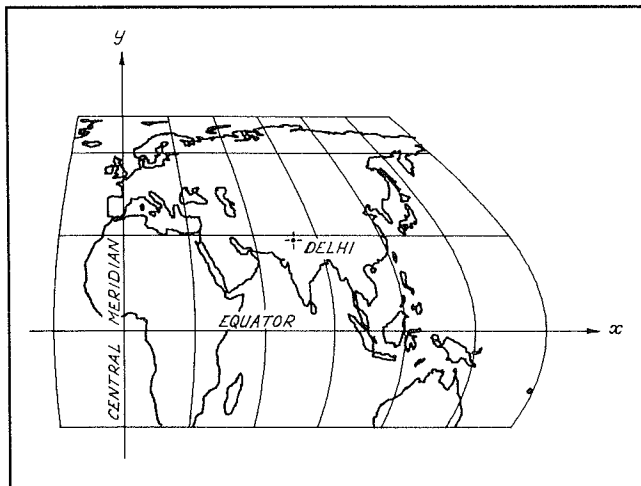
The partial differentials given in equations (9) are obtained in the following manner:



**Figure 10.** Minimum-error equal-area sinusoidal pseudocylindrical projection (axes ratio = 1/2; pole/equator ratio = 1/1.58; graticule interval, 30 degrees).



**Figure 11.** Tissot's indicatrix (axes ratio = 1/2; pole/equator ratio = 1/1.58).



**Figure A1.** Minimum-error equal-area sinusoidal pseudocylindrical projection (axes ratio = 1/2; pole/equator ratio = 1/1.58; graticule interval, 30 degrees).

From equations (6)

$$x = \frac{R \lambda}{M n} (k + \cos \alpha)$$

$$y = R M \alpha$$

$$n \sin \phi = k \alpha + \sin \alpha.$$

The last equation can be differentiated implicitly to give

$$n \cos \phi \frac{\partial \phi}{\partial \alpha} = k + \cos \alpha$$

hence,

$$\frac{\partial \alpha}{\partial \phi} = \frac{n \cos \phi}{k + \cos \alpha}.$$

Using the chain rule for differentiation gives the partial derivatives with respect to  $\phi$  as:

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= \frac{\partial x}{\partial \alpha} \frac{\partial \alpha}{\partial \phi} \\ &= \frac{-R \lambda \sin \alpha}{M n} \frac{n \cos \phi}{k + \cos \alpha} \\ &= \frac{-R \lambda \cos \phi \sin \alpha}{M (k + \cos \alpha)}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial y}{\partial \phi} &= \frac{\partial y}{\partial \alpha} \frac{\partial \alpha}{\partial \phi} \\ &= R M \frac{n \cos \phi}{k + \cos \alpha}, \end{aligned}$$

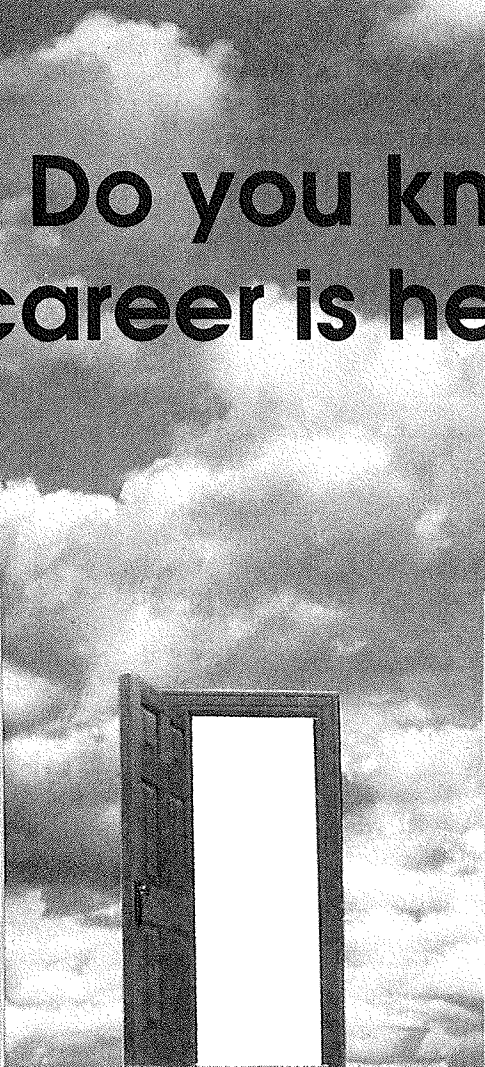
and the partial derivatives with respect to  $\lambda$  are

$$\frac{\partial x}{\partial \lambda} = \frac{R (k + \cos \alpha)}{M n} \text{ and } \frac{\partial y}{\partial \lambda} = 0.$$

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